# April 23

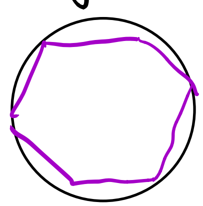
# Recap

- **Ruler**: Can draw a straight line between any 2 points
- **Compass** Can draw a circle centred around a given point & given radius

**Ques**: Which shapes can be constructed using these 2 operations?

Last time
- Bisect a line segment
- Trisect a line segment

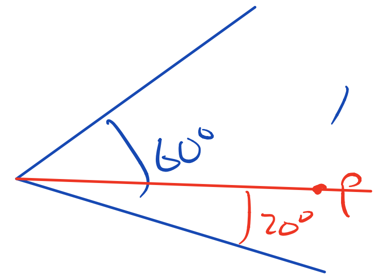
In HW,
- double a square
- bisect an angle
- inscribe a regular hexagon in a circle



**Goal**: Show it is impossible to:

① Trisect a $60^\circ$ angle

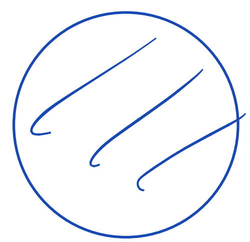
Given

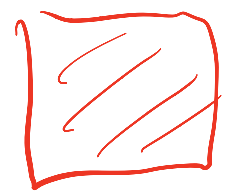



, cannot find another point P dividing angle into 3 parts

② Square a circle

Given a circle, cannot construct a square

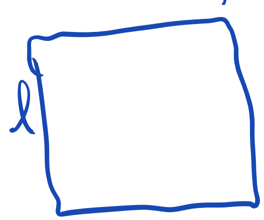




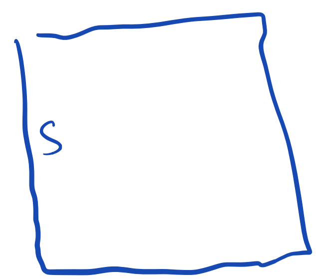
of same area.

③ Double a cube

Given a square, cannot construct another square




such that $s^3 = 2l^3$



Suppose we have points in $\mathbb{R}^2$
$p_1=(a_1,b_1)$, $p_2=(a_2,b_2)$, ..., $p_m=(a_m,b_m)$

Let $q_1=(x_1,y_1)$, $q_2=(x_2,y_2)$, ..., $q_n=(x_n,y_n)$
where each $q_i$ is constructed in one step from $p_1,\dots,p_m,q_1,\dots,q_{i-1}$
using ruler & compass.

Define field extensions

$$\begin{aligned}
&\mathbb{Q}\\
&\cap\\
&K_0=\mathbb{Q}(a_1,\dots,a_m,b_1,\dots,b_m)\subset\mathbb{R}\\
&\cap\\
&K_1=K_0(x_1,y_1)\\
&\cap\\
&K_2=K_1(x_2,y_2)\\
&\vdots\\
&K_n=K_{n-1}(x_n,y_n)\subset\mathbb{R}
\end{aligned}$$

**Prop:** Each $x_{i+1}$ and $y_{i+1}$ are roots of quadratic polynomials with coefficients in $K_i$.

**Cor:**

- $|K_{i+1}:K_i|=1,2$ or $4$
- $|K_n:K_0|=$ power of $2$

This implies all the new points constructed lie in $K_n$ and so their coordinates are algebraic over $K_0$ and their min polys have degree a power of 2.

This will give us contradictions!

**Prop:** Each $x_{i+1}$ and $y_{i+1}$ are roots of quadratic polynomials with coefficients in $K_i$.

**Cor:**

- $|K_{i+1} : K_i| = 1, 2$ or $4$
- $|K_n : K_0|$ = power of $2$

**Pf of Prop** Three cases to consider

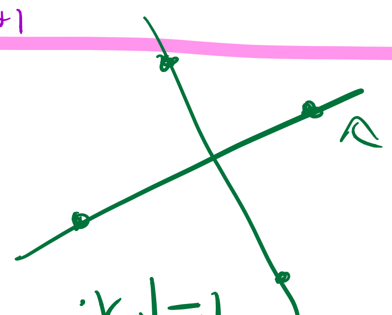
(Recall $K_{i+1} = K_i(x_{i+1}, y_{i+1})$ where $q_{i+1} = (x_{i+1}, y_{i+1})$ is new pt)

① $q_{i+1}$ is intersection of 2 lines ✓

② $q_{i+1}$ " " of 2 circles

③ $q_{i+1}$ " " of line & circle

①

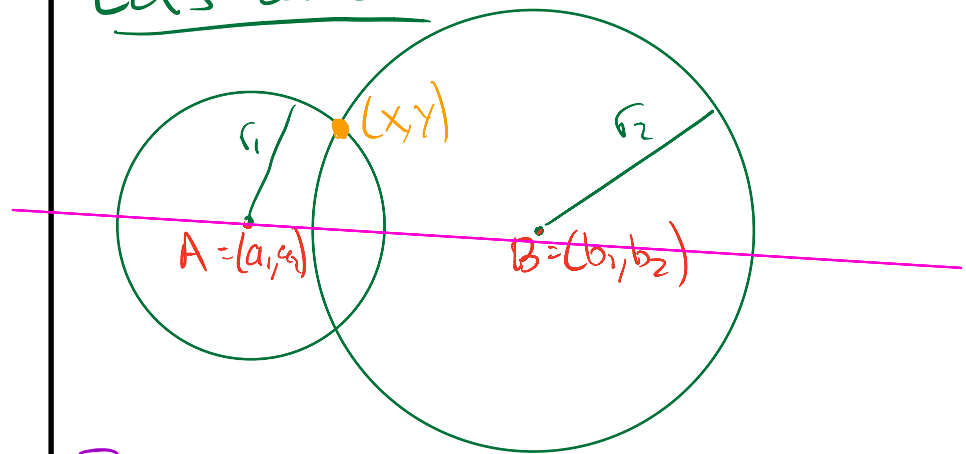


If all coordinates of these 4 pts lie in $K_i$, then intersection pt lies in $K_i$

$\Rightarrow |K_{i+1} : K_i| = 1$

Let's skip 3 (similar to 2)

Let's do 2




Eqns:

$$(x-a_1)^2 + (y-a_2)^2 = r_1^2$$
$$(x-b_1)^2 + (y-b_2)^2 = r_2^2$$

Rotate line so it's horizontal

(slope line $= \frac{b_2-a_2}{b_1-a_1} \in K_i$)

Can assume $a_2 = b_2$

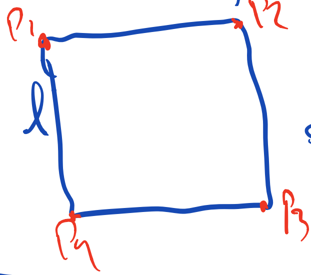
$$(y-a_2)^2 = r_1^2 - (x-a_1)^2 = r_2^2 - (x-b_1)^2$$

$\Rightarrow$ $x$ is solution to a quadratic

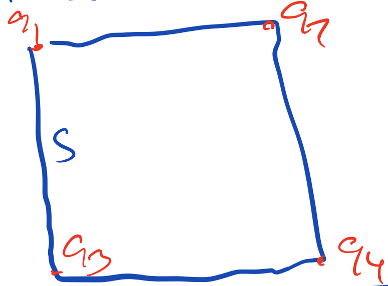
Same for $y$.

## ③ Double a cube

Given a square, cannot construct another square

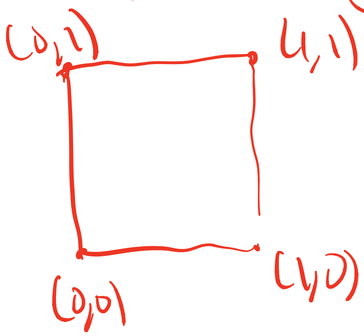



such that $s^3 = 2\ell^3$

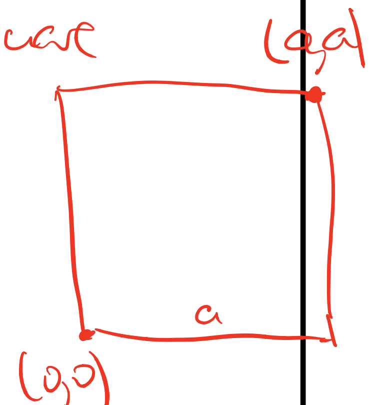



**Thm** (Wantzel 1837)
It's impossible to double cube.

**Pf:** Assume original square




suppose $\rightsquigarrow$




such that $a^3 = 2$

But $f(x) = x^3 - 2$ irred of degree 3

$$\rightsquigarrow K_0 = \mathbb{Q} \subset K_1 \subset \cdots \subset K_n \ni a$$

On one hand,

$|K_n : K_0|$ power of 2

On other hand,

$$\mathbb{Q} = K_0 \subset K_0(a) \subset K_n$$

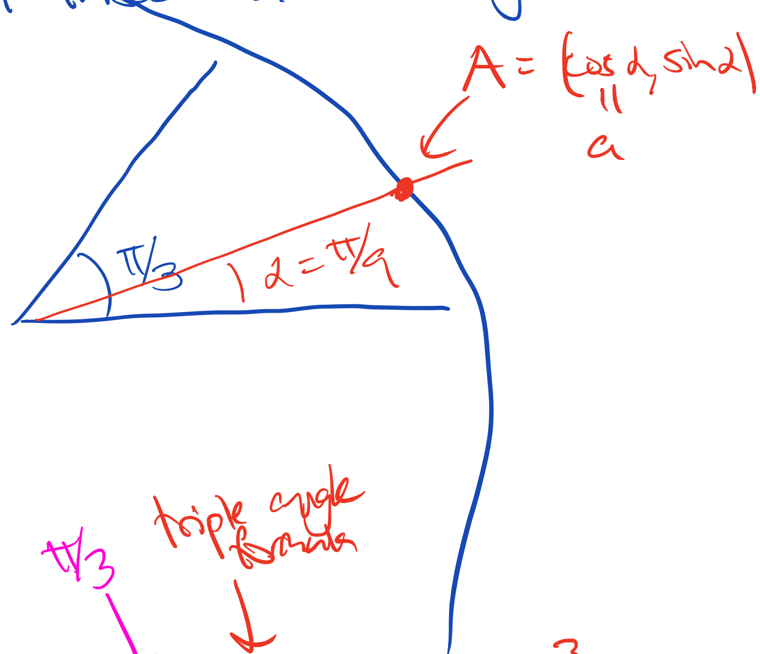
$\Rightarrow |\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}|$ power of 2

contradiction.

## Thm (Wantzel 1837)

Cannot trisect a $60^\circ$ angle.

Proof




$\pi/3$ triple angle formula

$$\frac{1}{2} = \cos(3\alpha) = 4\cos(\alpha)^3 - 3\cos(\alpha)$$

$\Rightarrow$ Therefore $a = \cos(\alpha)$ is a root of the poly

$f(x) = 4x^3 - 3x - \frac{1}{2}$ is irred$/\mathbb{Q}$

(why?)

$\Rightarrow [\mathbb{Q}(a):\mathbb{Q}] = 3$

So $a$ cannot be obtained by ruler & compass

**Thm** It is impossible to square the circle.

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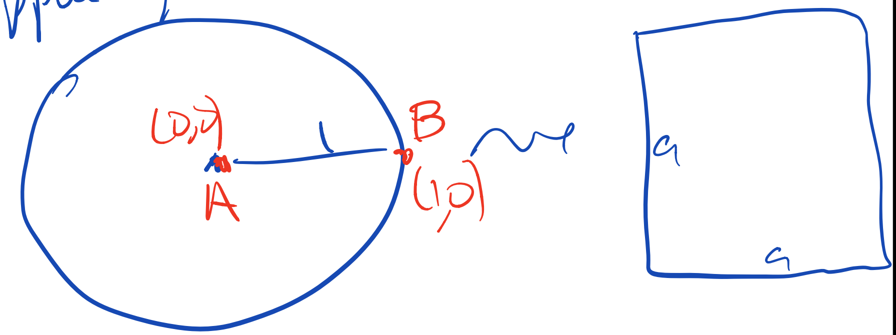
Suppose you can




Suppose $\exists$ n steps using ruler & compass given points $q_1, \ldots, q_n$

Necessarily, $a^2 = \pi$

$$a = \sqrt{\pi}$$

$$K_0 \subset K_1 \subset \cdots \subset K_n$$

$$\sqrt{\pi}$$

**Thm** $\pi$ is transcendental / $\mathbb{Q}$

It follows transcendentalness of $\pi$

$\Rightarrow$ $\sqrt{\pi}$ is not algebraic!

$\Rightarrow$ can't square circle.

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Is hard!

Archimedes $3\frac{1}{7} > \pi > 3\frac{10}{71}$

3.1429     3.1408

- $\sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.14153338$
- $\frac{355}{113} = 3.141592920$
- $\sqrt[4]{\frac{2143}{22}} = 3.141592652582\ldots$